Lecture 8:

02-10-18

§ Non-homogeneous eqt:

Def:

$$\boxed{y'' + p(t)y' + q(t)y = r(t)}$$

- If $r(t) = 0$, it is called homogeneous
- Otherwise, we said it is Non-homogeneous

Prop: If y_1 satisfy $y_1'' + p(t)y_1' + q(t)y_1 = r_1(t)$

y_2 satisfy $y_2'' + p(t)y_2' + q(t)y_2 = r_2(t)$.

then $y = y_1 + y_2$ satisfy $y'' + p(t)y' + q(t)y = r_1(t) + r_2(t)$

Pf: Exercise

Thm: Any solution y to the eqt: $y'' + p(t)y' + q(t)y = r(t)$ ---- $(*)$

can be written as

$$\boxed{y = c_1 y_1 + c_2 y_2 + Y(t)}$$

where $Y(t)$ is a solution to $(*)$, and y_1, y_2 fundamental set of solution to $y'' + p(t)y' + q(t)y = 0$.

Pf: Assume $y(t)$ is a solution to $(*)$

Then we consider $g(t) = y(t) - Y(t)$, by the about Proposition, we have $g(t)$ satisfy

$$g'' + p(t)g' + q(t)g = 0$$

by result from previous Lectures:

$$g = C_1 y_1 + C_2 y_2$$

$$\Rightarrow \quad y(t) = C_1 y_1(t) + C_2 y_2(t) + Y(t). \quad \boxtimes$$

Moral: All we need to find is $y_1(t)$, $y_2(t)$ and $Y(t)$.

§ Non-homogeneous eqt with constant coeff:

We will consider

$$\boxed{ay'' + by' + cy = r(t)}$$

Example: $\quad y'' - 3y' - 4y = 3e^{2t}.$

Consider homogeneous eqt:

$$y'' - 3y' - 4y = 0$$

Char eqt: $r^2 - 3r - 4 = 0$

$\Rightarrow \quad (r-4)(r+1) = 0$

hence $y_1 = e^{4t}$, $y_2 = e^{-t}$ is a fundamental set of solution to $y'' - 3y' - 4y = 0$

For particular sol $Y(t)$:

Guess: $Y(t) = Ae^{qt}$

Plug it into the required eqt: $Y'(t) = Aqe^{qt}$
$Y''(t) = Aq^2e^{qt}$

Need:

$$A(q^2 - 3q - 4)e^{qt} = 3e^{2t}.$$

ie. $q = 2$ and $A(4-6-4) = 3 \Rightarrow A = \frac{-1}{2}$

$\therefore \quad Y(t) = \frac{-1}{2}e^{2t}.$

General solution: $y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}$

Example: $y'' - 3y' - 4y = 2\sin(t).$

We guess: $Y(t) = a\sin(t) + b\cos(t).$

$$Y'(t) = a\cos(t) - b\sin(t)$$
$$Y''(t) = -a\sin(t) - b\cos(t).$$

$\therefore$ We need:

$$(-a+3b-4a)\sin(t) + (-b-3a-4b)\cos(t) = 2\sin(t).$$

$$\Rightarrow \quad -5b-3a=0 \Rightarrow b = \frac{3}{5}a$$

and $\quad -5a+3b = 2 \Rightarrow \frac{9}{5}a - 5a = 2$

$$\Rightarrow \frac{-16}{5}a = 2 \Rightarrow a = \frac{-5}{8}$$

General solution:

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{5}{8}\sin(t) + \frac{3}{5}\cos(t).$$

Rk: • If we let $Y(t) = a\sin(t)$ only, we will find that there is NO solution to the eqt. That means our guess is NOT correct and we have to modify it.

Example: $\quad y'' - 3y' - 4y = 8t^2 - 9$

Guess: $\quad Y(t) = At^2 + Bt + C$

$$Y'(t) = 2At + B$$
$$Y''(t) = 2A.$$

$\therefore$ require: $\quad 2A - 3(2At+B) - 4(At^2+Bt+C) = 8t^2 - 9$

$$\Rightarrow \quad -4At^2 - (4B+6A)t + (2A-3B-4C) = 8t^2 - 9$$

$$\Rightarrow \quad A = -2, \quad B = 3, \quad -4-9-4C = -9$$

$$C = -1.$$

$$\therefore \quad Y(t) = -2t^2 + 3t - 1.$$

and General solution:

$$y(t) = C_1 e^{4t} + C_2 e^{-t} + (-2t^2 + 3t - 1).$$

Example: $y'' - 3y' - 4y = e^{-t}$

Again we try $Y(t) = Ae^{-t}$

we compute $Y'(t) = -Ae^{-t}$, $Y''(t) = Ae^{-t}$

$\therefore$ we find $Y'' - 3Y' - 4Y = 0$, and it does NOT satisfy the equation!

Reason: e^{-t} is a solution to the homogeneous eqt

we need another trial.

Guess: $Y(t) = A(t)e^{-t}$ and see what happen.

$$Y'(t) = (A' - A)e^{-t}, \quad Y'' = (A'' - 2A' + A)e^{-t}$$

Plug into the equation we have

$$\left[\left(A'' - 2A' + \cancelto{0}{A}\right) - 3\left(A' - \cancelto{0}{A}\right) - \cancelto{0}{4A}\right] e^{-t} = e^{-t}$$

because e^{-t} is a solution to original eqt.

$$\Rightarrow \quad A'' - 5A' = 1.$$

$$\Rightarrow \quad \left(e^{-5t}A'\right)' = e^{-5t}$$

$$\Rightarrow \quad e^{-5t}A' = \frac{-1}{5}e^{-5t} + d_1$$

$$\Rightarrow \quad A' = \frac{-1}{5} + d_1 e^{5t}$$

$$\Rightarrow \quad A = \frac{-1}{5}t + d_1 e^{5t} + d_2$$

<u>Now observe that:</u>

We have $Y(t) = \frac{-1}{5} t e^{-t} + \cancelto{0}{\boxed{d_1 e^{4t} + d_2 e^{-t}}}$.

solution to homogeneous eqt, which is redundant!

<u>General solution:</u>

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{5} t e^{-t}$$

<u>Rk:</u> We have use the above technique before, which is known as the reduction of order!

Summary: $ay'' + by' + cy = r(t)$, the first thing to try is:

$r(t)$	Trial: $Y(t)$
1. $e^{\alpha t}$, which is NOT a solution to homo. eqt.	$Ae^{\alpha t}$
2. $e^{\alpha t}$, which is a solution to homo. eqt.	$(At^2 + Bt)e^{\alpha t}$ or $A(t)e^{\alpha t}$
3. $A_0 + \cdots + A_k t^k$ polynomial of degree k	$B_0 + \cdots + B_\ell t^\ell$ polynomial of suitable degree ℓ.
4. $a\cos(\alpha t) + b\sin(\beta t)$	$A\cos(\alpha t) + B\sin(\beta t)$

Less explore this situation:

suppose we want to solve it with polynomial with degree $\ell = k$:

$$\text{Let } Y(t) = B_0 + \cdots + B_\ell t^\ell, \text{ then}$$

$$Y'(t) = B_1 + \cdots + \ell B_\ell t^{\ell-1}$$

$$Y''(t) = 2B_2 + \cdots + \ell(\ell-1)B_\ell t^{\ell-2}$$

$$\Rightarrow (2aB_2 + bB_1 + cB_0) = A_0$$

$$\vdots$$

$$(a(j+2)(j+1)B_{j+2} + b(j+1)B_{j+1} + cB_j) = A_j$$

If we write $Y(t) = B_0 + B_1 t + \cdots + B_j t^j + \cdots$

$$= \sum_{j=0}^{\infty} B_j t^j \quad \text{with } B_j = 0 \text{ for } j > \ell$$

$$r(t) = A_0 + A_1 t^1 + \cdots + A_j t^j + \cdots$$

$$= \sum_{j=0}^{\infty} A_j t^j \quad \text{with } A_j = 0 \text{ for } j > k.$$

then we can simple write

$$a(j+2)(j+1)B_{j+2} + b(j+1)B_{j+1} + cB_j = A_j$$

Example: $\quad r(t) = A_0 + A_1 t$

$$4aB_2 + bB_1 + cB_0 = A_0.$$
$$6aB_3 + 2bB_2 + cB_1 = A_1$$

$$\vdots$$

Case 1: Let say we try to set $\ell = k$, i.e. $B_j = 0$ for $j > 1$.

We left with $\quad \begin{cases} bB_1 + cB_0 = A_0 \\ cB_1 = A_1 \end{cases}$

which is solvable if $c \neq 0$.

Case 2: Say $c = 0$, $b \neq 0$, then it is impossible to impose $\ell = k$, we try $\ell = k+1$.

$$\begin{cases} 4aB_2 + bB_1 = A_0 \\ 2bB_2 = A_1 \end{cases}$$ solvable! with $B_0 = 0$

Case 3: if $b = c = 0$, then we have $\ell = k+2$

$$\begin{cases} 4aB_2 = A_0 \\ 12aB_3 = A_1 \end{cases}$$ solvable! with $B_0 = B_1 = 0$

In general:

Case 1: $c \neq 0$, we can solve it with $\ell = k$.

by starting with $cB_k = A_k \rightsquigarrow$ solve B_k

$b(k-1)B_k + cB_{k-1} = A_{k-1} \rightsquigarrow$ solve B_{k-1}

$ak(k-1)B_k + b(k-1)B_{k-1} + cB_{k-2} = A_{k-2} \rightsquigarrow$ solve B_{k-2}

$\vdots$

if B_j, B_{j-1} is solved

$aj(j-1)B_j + b(j-1)B_{j-1} + cB_{j-2} = A_{j-2} \rightsquigarrow$ solve B_{j-2}

or equivalently:

$$\begin{pmatrix} c & \ddots & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & ak(k-1) \\ & & & \ddots & b(k-1) \\ 0 & & & & c \end{pmatrix} \begin{pmatrix} B_0 \\ \vdots \\ B_k \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

Case 2: $c=0$, but $b\neq 0$, then we let $B_0=0$ and $l=k+1$

$$\begin{pmatrix} b & & \times \\ & \ddots & \\ 0 & & kb \end{pmatrix}\begin{pmatrix} B_1 \\ \vdots \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

invertible!

Case 3: Let $B_0=B_1=0$, with $l=k+2$.

$$\begin{pmatrix} 2a & & 0 \\ & \ddots & \\ 0 & & a(k+2)(k+1) \end{pmatrix}\begin{pmatrix} B_2 \\ \vdots \\ B_{k+2} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

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§ More complicated situation:

1) • $r(t) = P_k(t)e^{\alpha t}$ where $P_k(t) = A_0 + \dots + A_k t^k + O$

Trial: Let $Y(t) = e^{\alpha t} Q_l(t)$ where $Q_l(t) = B_0 + \dots + B_l t^l + O$

try to differentiate:

$$Y'(t) = e^{\alpha t}(\alpha Q_l(t) + Q_l'(t))$$
$$= e^{\alpha t}\left((\alpha B_0 + B_1) + (\alpha B_1 + 2B_2)t + \dots + (\alpha B_j + jB_{j+1})t^j + \dots\right.$$

$$Y''(t) = e^{\alpha t}(\alpha^2 Q_l(t) + 2\alpha Q_l'(t) + Q_l''(t))$$

$$= e^{\alpha t}\Big((\alpha^2 B_0 + 2\alpha B_1 + 2B_2) + \cdots$$
$$+ \big(\alpha^2 B_j + 2\alpha(j+1) B_{j+1} + (j+2)(j+1) B_{j+2}\big)t^j + \cdots \Big)$$

Equation to solve:

1) $a(\alpha^2 B_0 + 2\alpha B_1 + 2B_2) + b(\alpha B_0 + B_1) + cB_0 = A_0$

$\Leftrightarrow (a\alpha^2 + b\alpha + c)B_0 + (2a\alpha + b)B_1 + 2aB_2 = A_0$.

2) $a(\alpha^2 B_1 + 4\alpha B_2 + 6B_3) + b(\alpha B_1 + 2B_2) + cB_1 = A_1$

$\Leftrightarrow (a\alpha^2 + b\alpha + c)B_1 + (4a\alpha + 2b)B_2 + 6aB_3 = A_1$

3) $a(\alpha^2 B_2 + 6\alpha B_3 + 12B_4) + b(\alpha B_2 + 3B_3) + cB_2 = A_2$

$\Leftrightarrow (a\alpha^2 + b\alpha + c)B_2 + (6a\alpha + 3b)B_3 + 12aB_4 = A_2$

4) $(a\alpha^2 + b\alpha + c)B_3 + 4(2a\alpha + b)B_4 + 20aB_5 = A_3$

General eq: Let $f(r) = ar^2 + br + c$

$$a(j+2)(j+1)B_{j+2} + (j+1)f'(\alpha)B_{j+1} + f(\alpha)B_j = A_j.$$

Moral: Notice that it is linear equation of the same type as our previous case by letting $\tilde{a}_{j+2} = (j+2)(j+1)a$, $\tilde{b}_{j+1} = (j+1)f'(\alpha)$, $\tilde{c} = f(\alpha)$.

Case i) If $\tilde{c} = f(\alpha) \neq 0$ (i.e. α NOT a root of $f(r)=0$)

- We can solve $Q_\ell(t)$ with $\ell = k$, via the eqt

$$\begin{pmatrix} \tilde{c} & \tilde{b}_1 & \hat{a}_2 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & & \tilde{b}_{k-1} & \hat{a}_k \\ & & & \tilde{c} & \tilde{b}_k \\ 0 & & & & \tilde{c} \end{pmatrix} \begin{pmatrix} B_0 \\ \vdots \\ B_k \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

Case ii) If $\tilde{c} = 0$, $\tilde{b}_j \neq 0$

(i.e. $f(\alpha) = 0$ with $f'(\alpha) \neq 0$ which means α is a simple root of $f(r)=0$)

We can solve it with $\ell = k+1$, and

$$\begin{pmatrix} \tilde{b}_1 & \hat{a}_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \hat{a}_{k+1} \\ 0 & & & \tilde{b}_{k+1} \end{pmatrix} \begin{pmatrix} B_1 \\ \vdots \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

Case iii) When $\tilde{c} = f(\alpha) = 0$, $\tilde{b}_j = (j+1)f'(\alpha) = 0$

i.e. α double root of $f(r)=0$

$$\rightsquigarrow \begin{pmatrix} \hat{a}_2 & & 0 \\ & \ddots & \\ 0 & & \hat{a}_{k+2} \end{pmatrix} \begin{pmatrix} B_2 \\ \vdots \\ B_{k+2} \end{pmatrix} = \begin{pmatrix} A_0 \\ \vdots \\ A_k \end{pmatrix}$$

$\boxtimes$

2) $r(t) = e^{\alpha t}\cos\beta t\, P_k(t)$ or $e^{\alpha t}\sin\beta t\, P_k(t)$.

A possible guess:

$$Y(t) = e^{\alpha t}\left(Q_\ell(t)\cos\beta t + R_\ell(t)\sin\beta t\right)$$

Reasoning: we write $e^{\alpha t}\cos\beta t = \frac{1}{2}\left(e^{\gamma t} + e^{\bar{\gamma} t}\right)$

$e^{\alpha t}\sin\beta t = \frac{1}{2i}\left(e^{\gamma t} - e^{\bar{\gamma} t}\right)$

and think about rewrite

$$r(t) = e^{rt} P_k(t) + e^{\bar{r}t} P_k(t)$$

$$\text{or } i\left(e^{rt} P_k(t) - e^{\bar{r}t} P_k(t)\right)$$

Moral: We only have to consider

$r(t) = e^{rt} P_k(t)$.

and look for $Y(t)$ of the form $e^{rt} Q_\ell(t)$

with some $Q_\ell(t)$ complex-valued.

Equation to solve:

$$a(j+2)(j+1)B_{j+2} + (j+1)f'(\gamma)B_{j+1} + f(\gamma)B_j = A_j.$$

Two possibility:

1) If $f(\gamma) \neq 0$, $\rightsquigarrow$ solve $Q_\ell(t)$ with $\ell = k$

2) If $f(\gamma) = 0$ but $f'(\gamma) \neq 0 \rightsquigarrow \exists\, Q_\ell(t)$ with $\ell = k+1$.